CALCULATION OF TEMPERATURE FIELDS IN DOMAINS WITH PIECEWISE-HOMOGENEOUS MEDIA JOINED ALONG A NON-CANONICAL BOUNDARY

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An analytical method is developed for the solution of linear problems of stationary heat conduction in domains with piecewise-homogeneous media joined along a non-canonical boundary. The method is based on the addition of non-canonical contours to contours described in the framework of classical systems of coordinates.

The use of various modifications of the method of partial domains to solve problems of stationary heat conduction assumes that a boundary joining mixed subdomains is canonical [1]. In the present paper we present an approximate analytical method that allows us to remove this condition.

With no loss of generality in our discussion, we consider the essence of our method through an example.

Assume that it is necessary to solve Laplace's equation in a domain consisting of a set of two bodies in contact (Fig. 1):

$$\nabla^2 T_i = 0,\ i = 1,\ 2, \tag{1}$$

and to satisfy the boundary conditions

$$T_1|_{y=0} = 0, \tag{2}$$

$$\left.\frac{\partial T_i}{\partial x}\right|_{x=0} = 0,\ i = 1,\ 2, \tag{3}$$

$$\left.\left(\frac{\partial T_i}{\partial x} + \frac{\alpha_i}{\lambda_i} T_i\right)\right|_{x=a} = 0,\ \ i = 1,\ 2, \tag{4}$$

$$T_2|_{y=b} = T_0, \tag{5}$$

$$(T_1 - T_2)|_\Gamma = 0, \tag{6}$$

$$\lambda_1 \left.\left(\frac{\partial T_1}{\partial x}\cos(\beta) + \frac{\partial T_1}{\partial y}\sin(\beta)\right)\right|_\Gamma = \lambda_2 \left.\left(\frac{\partial T_2}{\partial x}\cos(\beta) + \frac{\partial T_2}{\partial y}\sin(\beta)\right)\right|_\Gamma, \tag{7}$$

where Γ is the boundary of adjoining subdomains, defined by the equation

$$y = \frac{d+c}{2} + \frac{d-c}{2}\cos\left(\frac{\pi}{a}x\right),\ x \in [0,\ a];$$

β is the angle between the normal to the junction boundary and the ox-axis, defined in the given case by the relationship [2]:

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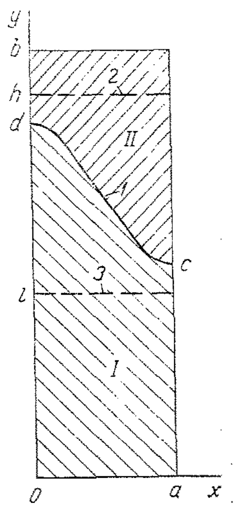

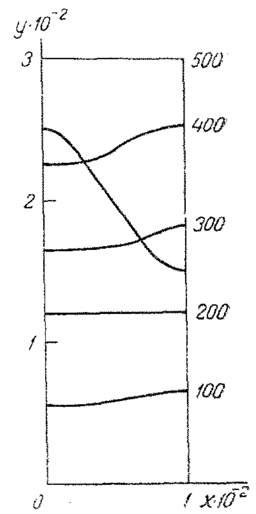

Fig. 1 Fig. 2

Fig. 1. Form of computational domains: 1) non-canonical boundary joining subdomains I and II; 2) boundary of subdomain I constructed to canonical form; 3) boundary of subdomain II constructed to canonical form; a, b, c, d, ℓ, h) coordinates of nodes of corresponding subdomains.

Fig. 2. Temperature distribution in computational domain.

$$\beta = \beta(x) = \frac{\pi}{2} - \operatorname{arctg}\left(\frac{c-d}{2a}\,\pi \sin\left(\frac{\pi}{a}x\right)\right).$$

To solve problem (1)-(7), as a preliminary we construct a contour bounding subdomains to canonical form. Subdomain 1 is bounded by the set of intervals: $y = 0$, $x \in [0, z]$; $y = h$, $x = [0, a]$; $x = 0$, $y \in [0, h]$; $x = a$, $y \in [0, h]$. Subdomain 2 is bounded by the set of intervals: $y = \ell$, $x \in [0, a]$; $y = b$, $x \in [0, a]$; $x = 0$, $y \in [\ell, b]$; $x = a$, $y \in [\ell, b]$.

On the added portions of the contours ($y = h$ and $y = \ell$) we introduce the auxiliary boundary conditions

$$\lambda_1 \frac{\partial T_1}{\partial y}\bigg|_{y=h} = q_j,\quad x \in \left((j-1)\frac{a}{M},\ ja/M\right),\ j = 1, 2, \ldots, M, \tag{8}$$

$$\lambda_2 \frac{\partial T_2}{\partial y}\bigg|_{y=l} = f_j,\quad x \in \left((j-1)\frac{a}{M},\ ja/M\right),\ j = 1, 2, \ldots, M. \tag{9}$$

In the expanded domain 1 we seek a solution of Laplace's equation with boundary conditions (3) $x = 0$, $y \in [0, h]$, boundary conditions (4) on the boundary $x = a$, $y \in [0, h]$ boundary condition (2) on the boundary $y = 0$, $x \in [0, a]$ and boundary condition (8) on the boundary $y = h$, $x \in [0, a]$.

The solution of the first auxiliary problem constructed by the method of separation of variables [3] has the form

$$T_1(x, y, q_i, i = 1, 2, \ldots, M) = \sum_{k=1}^{\infty} A_k \operatorname{sh}(\omega_k y)\cos(\omega_k x), \tag{10}$$

where

$$A_k = \frac{8\sin\left(\frac{a}{2M}\omega_k\right)\sum_{i=1}^{M} q_i \cos\left(\frac{a}{2M}(2i-1)\,\omega_k\right)}{(2a\omega_k + \sin(2\omega_k a))\,\lambda_1\,\mathrm{ch}(\omega_k h)};$$

ω_k are the roots of the transcendental equation $\mathrm{tg}(\omega_k a) = \alpha_1/(\lambda_1\omega_k)$.

In the expanded domain 2 we use the method of separation of variables to solve Laplace's equation with the boundary conditions (3) on the boundary $x = 0$, $y \in [\ell, b]$, and condition (9) on boundary $y = \ell$, $x \in [0, a]$, condition (5) on boundary $y = b$, $x \in [0, a]$:

$$T_2(x, y, f_i, i = 1, 2, \ldots, M) = \sum_{k=1}^{\infty} (B_k\,\mathrm{ch}(\mu_k(y - l)) + C_k\,\mathrm{sh}(\mu_k(b - y)))\cos(\mu_k x), \tag{11}$$

where

$$B_k = \frac{4T_0\sin(\mu_k a)}{(2a\mu_k + \sin(2\mu_k a))\,\lambda_2\,\mathrm{ch}(\mu_k(b - l))};$$

$$C_k = \frac{8\sin\left(\frac{a}{2M}\mu_k\right)\sum_{i=1}^{M} f_i\cos\left(\frac{a}{2M}(2i-1)\,\mu_k\right)}{(2a\mu_k + \sin(2\mu_k a))\,\lambda_2\,\mathrm{ch}(\mu_k(b - l))}; \tag{12}$$

μ_k are the roots of the transcendental equation

$$\mathrm{tg}(\mu_k a) = \frac{\alpha_2}{\lambda_2\mu_k}. \tag{13}$$

Since the value of the temperature at the i-th point belonging to the junction boundary of the subdomains can be found from the solution of the field problem, both for the 1st and for the 2nd subdomain, we can write

$$\sum_{j=1}^{M} (a_{ij}q_j - b_{ij}f_j) = F_i, \quad i = 1, 2, \ldots, M, \tag{14}$$

where, in accordance with [3],

$$a_{ij} = T_1\left(x_i,\ y_i = \frac{d+c}{2} + \frac{d-c}{2}\cos\left(\frac{\pi}{a}x_i\right),\right.$$

$$\left. q_1 = \ldots = q_{j-1} = q_{j+1} = \ldots = q_M = 0,\ q_j = 1\right),$$

$$b_{ij} = T_2\left(x_i,\ y_i = \frac{d+c}{2} + \frac{d-c}{2}\cos\left(\frac{\pi}{a}x_i\right),\right.$$

$$\left. T_0 = 0,\ f_1 = \ldots = f_{j-1} = f_{j+1} = \ldots = f_M = 0,\ f_j = 1\right),$$

$$F_i = T_2\left(x_i,\ y_i = \frac{d+c}{2} + \frac{d-c}{2}\cos\left(\frac{\pi}{a}x_i\right),\ f_j = 0,\ j = 1, 2, \ldots, M\right).$$

The system of equations (14) replaces, in what follows, the boundary condition (6). To use boundary condition (7) we must first determine the components of the thermal flow density vector at nodes of the collocation by a direct differentiation of expressions (10) and (11) with respect to x and y:

$$\left.\frac{\partial T_1}{\partial x}\right|_{\Gamma} = -\sum_{k=1}^{\infty} A_k \omega_k \operatorname{sh}(\omega_k y_i) \sin(\omega_k x_i), \quad i = 1, \ldots, M, \tag{15}$$

$$\left.\frac{\partial T_1}{\partial y}\right|_{\Gamma} = \sum_{k=1}^{\infty} A_k \omega_k \operatorname{ch}(\omega_k y_i) \cos(\omega_k x_i), \quad i = 1, \ldots, M, \tag{16}$$

$$\left.\frac{\partial T_2}{\partial x}\right|_{\Gamma} = -\sum_{k=1}^{\infty} \mu_k (B_k \operatorname{ch}(\mu_k (y_i - l)) + C_k \operatorname{sh}(\mu_k (b - y_i))) \sin(\mu_k x_i), \tag{17}$$

$$\frac{\partial T_2}{\partial y} = \sum_{k=1}^{\infty} \mu_k (B_k \operatorname{sh}(\mu_k (y_i - l)) - C_k \operatorname{ch}(\mu_k (b - y_i))) \cos(\mu_k x_i). \tag{18}$$

Using the principle of superposition of thermal fields, valid also for thermal flows, we can write

$$\begin{gathered}\left.\frac{\partial T_1}{\partial x}\right|_{(x_i, y_i)} = \sum_{j=1}^{M} C_{ij} q_i, \quad \left.\frac{\partial T_2}{\partial x}\right|_{(x_i, y_i)} = \sum_{j=1}^{M} P_{ij} f_i + \Psi_1(x_i, y_i), \\ \left.\frac{\partial T_1}{\partial y}\right|_{(x_i, y_i)} = \sum_{j=1}^{M} d_{ij} q_i, \quad \left.\frac{\partial T_2}{\partial y}\right|_{(x_i, y_i)} = \sum_{j=1}^{M} S_{ij} f_i + \Psi_2(x_i, y_i), \quad i = 1, 2, \ldots, M,\end{gathered} \tag{19}$$

where

$$C_{ij} = \frac{\partial T_1}{\partial x}\left(x_i,\ y_i = \frac{d+c}{2} + \frac{d-c}{2}\cos\left(\frac{\pi}{a} x_i\right),\right.$$
$$\left. q_1 = \ldots = q_{j-1} = q_{j+1} = \ldots = q_M = 0,\ q_j = 1\right);$$

$$d_{ij} = \frac{\partial T_1}{\partial y}\left(x_i,\ y_i = \frac{d+c}{2} + \frac{d-c}{2}\cos\left(\frac{\pi}{a} x_i\right),\right.$$
$$\left. q_1 = \ldots = q_{j-1} = q_{j+1} = \ldots = q_M = 0,\ q_j = 1\right);$$

$$P_{ij} = \frac{\partial T_2}{\partial x}\left(x_i,\ y_i = \frac{d+c}{2} + \frac{d-c}{2}\cos\left(\frac{\pi}{a} x_i\right),\right.$$
$$\left. T_0 = 0,\ f_1 = \ldots = f_{j-1} = f_{j+1} = \ldots = f_M = 0,\ f_j = 1\right);$$

$$S_{ij} = \frac{\partial T_2}{\partial y}\left(x_i,\ y_i = \frac{d+c}{2} + \frac{d-c}{2}\cos\left(\frac{\pi}{a} x_i\right),\right.$$
$$\left. T_0 = 0,\ f_1 = \ldots = f_{j-1} = f_{j+1} = \ldots = f_M = 0,\ f_j = 1\right);$$

$$\Psi_1(x_1, y_i) = \frac{\partial T_2}{\partial x}\left(x_i,\ y_i = \frac{d+c}{2} + \frac{d-c}{2}\cos\left(\frac{\pi}{a} x_i\right),\right.$$
$$\left. f_j = 0,\ j = 1, 2, \ldots, M\right);$$

$$\Psi_2(x_i, y_i) = \frac{\partial T_2}{\partial y}\left(x_i,\ y_i = \frac{d+c}{2} + \frac{d-c}{2}\cos\left(\frac{\pi}{a} x_i\right),\right.$$
$$\left. f_j = 0,\ j = 1, 2, \ldots, M\right).$$

Substituting formulas (19) into relation (7), we obtain

$$\sum_{j=1}^{M} [\lambda_1 (C_{ij} \cos(\beta_i) + d_{ij} \sin(\beta_i))\, q_i - \lambda_2 (P_{ij} \cos(\beta_i) + S_{ij} \sin(\beta_i))\, f_i] = \\ = \lambda_2 [\Psi_1 (x_i, y_i) \cos(\beta_i) + \Psi_2 (x_i, y_i) \sin(\beta_i)], \; i = 1, 2, \ldots, M. \tag{20}$$

The set of equations (14) and (20) forms a system of linear algebraic equations in the unknowns q_i and f_i. Solution of this system by the method of Gauss [4] makes it possible to find q_i and f_i, the substitution of which into formulas (10) and (11) yields an approximate analytical solution of the initial problem (1)-(7).

We solved this problem on the EC-1045 computer for the following values of the parameters: $a = 10^{-2}$ m, $b = 3 \cdot 10^{-2}$ m, $d = 2.5 \cdot 10^{-2}$ m, $C = 1.5 \cdot 10^{-2}$ m, $h = 2.7 \cdot 10^{-2}$ m, $\ell = 1.3 \cdot 10^{-2}$ m, $\lambda_1 = 400$ W/(m·deg), $\lambda_2 = 100$ W/(m·deg), $\alpha_1 = 800$ W/(cm^2·deg), $\alpha_2 = 100$ W/(cm^2·deg), $M = 10$, $T_0 = 500$°C. Results of the calculation for the system temperature field are shown in Fig. 2.

The approximate analytical solution obtained for the problem satisfies Laplace's equation exactly in domains 1 and 2, as well as the boundary conditions on the exterior contour. On the boundary where the solutions are spliced together boundary conditions of the fourth kind are satisfied at 10 points of the collocation. The time to calculate temperatures at 400 points of the domain takes about one minute of machine time.

A comparison of the temperature calculated by the proposed method with the exact analytical solution for the case where the junction boundary is the line $y = (d + c)/2$, shows that the relative error in the calculated temperature is at most 1%.

NOTATION

λ_1, λ_2, thermal conductivity coefficients; α_1, α_2, heat emission coefficients; T_1, T_2, temperature distribution functions; 2M, number of "auxiliary" sources q_i and f_i on the added boundaries.

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